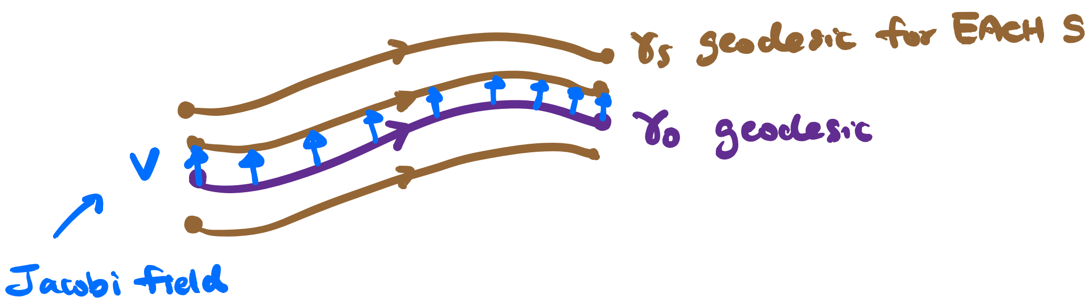
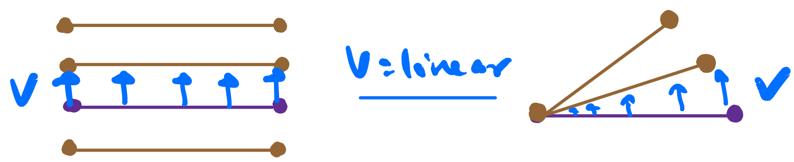
§ Jacobi Fields

Jacobi fields — variation field for family of geodesics
— "linearized" geodesic equations.
— kernel of "index form", i.e. Hessian of E

Motivations



E.g.) In $\mathbb{R}^2$



Def^{n}: Given a geodesic $\gamma(t) : [a,b] \to M$, a vector field $V(t)$, $t \in [a,b]$, along γ is said to be a Jacobi field if

$$(J) \qquad \boxed{\nabla_{\gamma'} \nabla_{\gamma'} V + R(\gamma', V)\gamma' = 0}$$

Jacobi field eq^{n}

Def^{n}: The index form of a geodesic $\gamma : [a,b] \to M$ is

$$I(V,W) := \int_a^b \langle \nabla_{\gamma'} V, \nabla_{\gamma'} W \rangle - \langle R(\gamma', V)\gamma', W \rangle \, dt$$

Note: $I(V,V) = E''(0)$ along the variation field V

Symmetry of $R \Rightarrow I(V,W)$ is a symmetric bilinear form.

Prop: Let V be a Jacobi field along a geodesic $\gamma : [a,b] \to M$.

THEN, $V \in$ "$\ker(I)$", ie

$$I(V,W) = 0 \quad \forall\, W \text{ v.f. along } \gamma \text{ st. } W(a) = 0 = W(b)$$

In fact, the converse also hold.

Proof: Recall the index form

$$I(V,W) := \int_a^b \langle \nabla_{\gamma'} V, \nabla_{\gamma'} W \rangle - \langle R(\gamma', V)\gamma', W \rangle \, dt$$

Integrate by part, using W vanishes at the end pts.

$$I(V,W) := \int_a^b - \langle \nabla_{\gamma'} \nabla_{\gamma'} V, W \rangle - \langle R(\gamma', V)\gamma', W \rangle \, dt$$

$$= - \int_a^b \langle \underbrace{\nabla_{\gamma'} \nabla_{\gamma'} V + R(\gamma', V)\gamma'}_{=0 \iff (J)}, W \rangle \, dt$$

$\blacksquare$

Suppose $\gamma_s : [a,b] \to M$ is a geodesic for EACH $s \in (-\varepsilon, \varepsilon)$

i.e. $\forall\, s \in (-\varepsilon, \varepsilon)$, $\boxed{\nabla_{\gamma_s'} \gamma_s' \equiv 0}$ ← non-linear 2nd order ODE system.

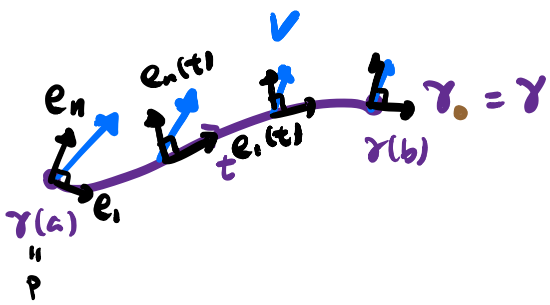
IDEA: If we differentiate the geodesic eq^{n} w.r.t. s at $s = 0$.

then we obtain the Jacobi field eq^{n} (J) for $V := \left.\dfrac{\partial \gamma}{\partial s}\right|_{s=0}$.

(Ex: Prove this!)

Lemma: (J) is a 2^{nd} order LINEAR ODE system.

Proof:



Fix O.N.B. $\{e_1, \dots, e_n\}$ of T_pM

Parallel transport along γ

$\Rightarrow$ obtain O.N.B. $\{e_1(t), \dots, e_n(t)\}$ for $T_{\gamma(t)}M$

Write $V(t) = \sum_{i=1}^{n} a_i(t)\, e_i(t)$. ← parallel

for some fcn $a_i(t)$.

(J) $\nabla_{\gamma'}\nabla_{\gamma'} V + R(\gamma', V)\gamma' = 0$

$$\Leftrightarrow \quad \sum_{i=1}^{n} a_i''(t)\, e_i(t) + \sum_{i=1}^{n} a_i(t)\, \underbrace{R(\gamma', e_i(t))\gamma'}_{\sum_{j=1}^{n} \langle R(\gamma', e_i(t))\gamma', e_j(t)\rangle e_j(t)} = 0$$

$$\Leftrightarrow \quad \underbrace{a_i''(t) + \sum_{j=1}^{n} a_j(t)\, R(\gamma', e_j, \gamma', e_i) = 0}_{2^{nd} \text{ order linear system in } a_i(t)} \qquad \forall i$$

Cor: (J) is uniquely solvable on $[a,b]$ for any given initial data $V(a)$ and $V'(a) := (\nabla_{\gamma'} V)(a)$.

depends linearly on initial data

Note that: Any vector field V along γ decompose:

$$V = V^{T} + V^{\perp}$$

tangent to γ — normal to γ

Prop: Any tangential (i.e. $V^\perp \equiv 0$) Jacobi field V along γ has the form

$$V(t) = \underbrace{\left(A + B(t-a)\right)}_{\text{linear in } t} \gamma'(t) \quad \text{for some constants } A, B \in \mathbb{R}$$

Pf: Solving uniquely (J) with initial data

$$V(a) = A\,\gamma'(a) \quad \text{and} \quad V'(a) = B\,\gamma'(a).$$

$\blacksquare$

Remark: This implies "tangential" Jacobi fields are NOT useful, but the "normal" Jacobi fields contains a lot of information about the geometry of (M^n, g).

Prop: Suppose $\gamma_s : [a,b] \to M$ is a 1-parameter family s.t. γ_s is a geodesic for EACH $s \in (-\varepsilon, \varepsilon)$.

THEN, the variation field $V := \left.\dfrac{\partial \gamma}{\partial s}\right|_{s=0}$ satisfies (J).

Remark: The converse is also true, i.e. Jacobi fields along geodesics are all "integrable". (Pf. HW)

Proof: Each γ_s is a geodesic

$$\Rightarrow \quad \nabla_{\gamma_s'} \gamma_s' \equiv 0 \quad \forall\, s \in (-\varepsilon, \varepsilon)$$

$$\Leftrightarrow \quad \boxed{\nabla_{\frac{\partial}{\partial t}} \frac{\partial \gamma}{\partial t} \equiv 0} \quad (*) \quad \forall\, s \in (-\varepsilon, \varepsilon).$$

Consider

$$\nabla_{\frac{\partial}{\partial t}} \nabla_{\frac{\partial}{\partial t}} \frac{\partial \gamma}{\partial s} \overset{\text{torsion-free}}{=} \nabla_{\frac{\partial}{\partial t}} \nabla_{\frac{\partial}{\partial s}} \frac{\partial \gamma}{\partial t}$$

$$= \nabla_{\frac{\partial}{\partial s}} \overbrace{\nabla_{\frac{\partial}{\partial t}} \frac{\partial \gamma}{\partial t}}^{(*)\,=\,0} + R\left(\frac{\partial \gamma}{\partial s}, \frac{\partial \gamma}{\partial t}\right) \frac{\partial \gamma}{\partial t}$$

Evaluate at $s = 0$,

$$\nabla_{\frac{\partial}{\partial t}} \nabla_{\frac{\partial}{\partial t}} V + R\left(\frac{\partial \gamma}{\partial t}, V\right) \frac{\partial \gamma}{\partial t} = 0 \qquad (J)$$

$\blacksquare$